BERNOULLI TWO-ARMED BANDITS

WITH GEOMETRIC TERMINATION

by

Donald A. Berry*

and

W. Kip Viscusi**

Technical Report No. 357

University of Minnesota

September 1979

*School of Statistics, University of Minnesota, Minneapolis, MN 55455. Research supported in part by the NSF under Grant No. 78-02694.

**Department of Economics, Northwestern University, Evanston, IL 60201.

ABSTRACT

The standard Bernoulli two-armed bandit model is modified by terminating the choice problem after the first unsuccessful trial. Both terminal reward situations and instances in which payoffs accrue with each success are considered. For independent machines, the stay-on-a-winner rule holds in each of these instances. Moreover, for the terminal payoff case, staying on a winner is optimal with interdependent machines. Increased prior information concerning the properties of a machine decreases its attractiveness by diminishing the prospect for long-term survival.

Key words and phrases: Two-armed bandit, Bernoulli bandit, stay-on-a-winner rule, geometric termination.

BERNOULLI TWO-ARMED BANDITS WITH GEOMETRIC TERMINATION

1. Introduction

In discrete-time two-armed bandit problems, one of two stochastic processes is selected at each of a number of stages. The process selected at a stage depends on the history of selections and results, so the decision problem is sequential, or dynamic. When the processes are Bernoulli, the usual objective is to maximize the expected number of successes, possibly discounted. Recent contributions to this problem include Fabius and van Zwet [5], Berry [1], Joshi [8], Gittins [7], and Berry and Fristedt [2], all of which contain additional references. Of special importance historically are papers by Thompson [9], Bradt, Karlin, and Johnson [3], and Feldman [6].

The problem considered here is a modified version of the two-armed bandit described above -- now the objective is to maximize the expected number of successes (again possibly discounted) before the first failure. This problem was considered by Viscusi [10] for the individual job choice problem involving uncertainties, where the worker may remain with a firm after a favorable outcome but must leave after an unfavorable outcome -- being fired, killed, or disabled, for example. Another possible application involves environmental management; Viscusi and Zeckhauser [11] discuss a Markovian decison problem which terminates when the process reaches an absorbing state.

The problem structure considered here is also encountered in analyses of medical treatments in which two drugs can be used for a particular disease, but only one drug at a time. A patient is to be treated with one of the drugs each week, say, until an unfavorable outcome (e.g., death) occurs, at which time treatment must stop. Most diseases are present in a variety of levels so the simplistic assumption made here of dichotomous responses is not always appropriate. Some of the uncertainty involved in a trial may be patient specific, so that, eventually, each trial begins with about the same kind of uncertainty, and learning takes place only within a trial. Our approach considers a particular trial with the information present initially suitably quantified, whatever its source may be.

In Section 3 we consider complete discounting for every stage but one, say the nth. This corresponds to the medical trial case in which the only objective is to keep the patient alive through n stages of treatment. The results of Section 3 hold for both dependent and independent processess.

In Section 4 we consider geometric discounting, both infinite and finite horizon. The processes are assumed to be independent for the analysis of Section 4.

The major result is the same for both Sections 3 and 4: namely, there is an optimal selection procedure under which the same process is observed at each stage. If such a procedure is followed, then, conditional on the parameter of the process, the time to termination has a geometric distribution.

These results hold, with evident modifications, for m machines, $m > 2$, as well as for two machines. The proofs given can easily be generalized; we present the case $m = 2$ for expository reasons.

We give a precise statement of the general problem in the next section. However, we shall avoid extensive notation and terminology. The interested reader is referred to Dubins and Savage [4] for a formal and extensive development of a general theory of gambling.

2. Statement of the General Problem

Let $(X_1,X_2,\ldots)$ and $(Y_1,Y_2,\ldots)$ be sequences of Bernoulli random variables generated by Machine 1 and Machine 2, respectively; let p_1 and p_2 denote the corresponding probabilities of the outcome "1". Given p_1 and p_2, $(X_1,X_2,\ldots)$ and $(Y_1,Y_2,\ldots)$ are assumed to be independent sequences of independent random variables. Both p_1 and p_2 are unknown; we take the Bayesian point of view and let $F(p_1,p_2)$ denote the (joint) distribution function, and also the distribution measure, of (p_1,p_2). The "information" present initially is then given by F. Expectation E will be with respect to F. Let F_1 and F_2 denote the marginal distribution functions (and measures) of p_1 and p_2.

The decision maker can choose to observe either X_k or Y_k at stage k; let W_k be the variable observed at k when following a particular strategy. The choice at stage k can depend on the previous $k - 1$ choices and on $W_1,\ldots,W_{k-1}$ -- but not on variables not chosen for observation. For $k = 1,2,\ldots,$ define

$$Z_k = \prod_{j=1}^{k} W_j.$$

Any sequence $\underset{\sim}{A} = (\alpha_1,\alpha_2,\ldots)$ with each $\alpha_k \geq 0$ is called a discount sequence. The decision maker's objective is to maximize his expected payoff:

$$E \sum_{1}^{\infty} \alpha_k Z_k,$$

where expectation is calculated for the strategy followed. Let

$$V^* = \sup E \sum_1^\infty \alpha_k Z_k,$$

where the supremum is over all possible strategies.

Every strategy that has expected payoff V^* is called optimal. We shall find every optimal strategy for the discount sequences considered in this paper. The basic tool in our demonstrations is "the fundamental theorem of gambling" (Dubins and Savage [4, Theorem 2.12.1]) in which a strategy is shown to be optimal by showing that its expected payoff is excessive.

3. Terminal Rewards Case

In this section we consider the discount sequence:

$$\underset{\sim}{A} = (0,\ldots,0,1,0,\ldots);$$

all the α_k are 0 except for one, say α_n, which is taken to be 1 without losing generality. The machines are used (or, processes observed) with the single objective of getting at least n immediate successes. The following theorem says that it is optimal to use either Machine 1 or Machine 2 exclusively for the entire trial (or, at least until a failure obtains or stage n is reached). That is, V^* is the maximum of the nth moments of F_1 and F_2.

Theorem 3.1. For a discount sequence with $\alpha_n = 1$ and $\alpha_k = 0$, $k \neq n$, and all initial distributions F, an optimal strategy is to use Machine i exlusively if $Ep_i^n = \max\{Ep_1^n, Ep_2^n\}$. Furthermore, it is uniquely optimal to use a single machine (i.e., never to switch) provided $P_F(p_1 = p_2) < 1$.

Remark. It is straightforward to prove this theorem by showing that $\max\{Ep_1^n, Ep_2^n\}$ is excessive. However, for this rather simple discount sequence a more direct proof is possible, and we present one.

Proof of Theorem 3.1. In general, Machine 1 can be used at k stages and Machine 2 at the remaining $n - k$ stages. Now, V^* is the maximal probability of n immediate successes:

$$V^* = \max_{0 \leq k \leq n} E(p_1^k p_2^{n-k}).$$

Regarding k as real rather than integral, we have

$$\frac{\partial^2}{\partial k^2} E(p_1^k p_2^{n-k}) = E\left(p_1^k p_2^{n-k}\left[\log p_1 - \log p_2\right]^2\right) \geq 0.$$

That is, $E(p_1^k p_2^{n-k})$ is convex in k and so attains its maximum at $k = 0$ or $k = n$, and the first result follows.

The uniqueness part of the theorem now follows from the fact that

$$V^* > \max_{1<k<n} E(p_1^k p_2^{n-k}) \quad \text{unless} \quad P_F(p_1 = p_2) = 1. \ \square$$

Example 3.1. Suppose the p_i have beta densities:

$$dF_i(x) \propto x^{a_i-1}(1-x)^{b_i-1}dx, \quad x \in (0,1),$$

with a_i, $b_i > 0$, $i = 1,2$ (where p_1 and p_2 may be dependent). Then

$$Ep_i^n = \frac{a_i}{a_i + b_i} \cdots \frac{a_i + n-1}{a_i + b_i + n-1}.$$

Without loss of generality assume $a_1 + b_1 \geq a_2 + b_2$, so that at least as much is known about Machine 1 as about Machine 2. Define

$$r(j) = \frac{a_1 + j-1}{a_1 + b_1 + j-1} \Big/ \frac{a_2 + j-1}{a_2 + b_2 + j-1}.$$

According to the theorem, Machine 1 is optimal if

$$(3.1) \qquad Ep_1^n/Ep_2^n = \prod_{j=1}^{n} r(j) \geq 1.$$

There is a number n^*, possibly infinite, such that (3.1) holds for all $n \geq n^*$. For, $r(n) = 1$ has exactly one real solution when $b_2 \neq b_1$; namely,

$$n_0 = \frac{a_2b_1-a_1b_2}{b_2-b_1} + 1,$$

which provides a lower bound for n^*.

Suppose $b_2 \geq b_1$. Then

$$a_1b_2 \geq a_1b_1 \geq (a_2 + b_2-b_1)b_1 = a_2b_1 + (b_2-b_1)b_1 \geq a_2b_1,$$

and $n_0 \leq 1$. That is, $r(n) \geq 1$ for all n, and, therefore, $n^* = 1$ and (3.1) holds for all n. The fact that the "prior number of failures" with Machine 2 is larger than with Machine 1 more that compensates for the fact that less is known about Machine 2.

Now suppose $b_1 < b_2$. Then n_0 is finite but may be <1, $=1$, or >1. If $n_0 \leq 1$ then $n^* = 1$. If $n_0 > 1$ then, as more generally, $n^* \geq n_0$. But n^* may be arbitrarily larger than n_0, and in fact may be infinite; that is, it may be that

$$\prod_{j=n_0}^{\infty} r(j) < 1/ \prod_{j=1}^{n_0-1} r(j)$$

even though $n_0 < \infty$. Still, the machine about which less is known represents a more desirable choice for larger values of n.

The monotoneity of the optimal strategy in n -- demonstrated here for beta distributions -- does not hold for arbitrary distributions, even though moment sequences for distributions on [0,1] are extremely regular (being completely monotone.□

Theorem 3.1 can be viewed as a <u>stay-on-a-winner rule</u>. For the classical two-armed bandit, Berry [1] shows that there is an optimal strategy that stays on a winner when the machines are independent and Bradt, Johnson, and Karlin [3] give a counterexample (for $\alpha_1 = \alpha_2 = 1$, $\alpha_k = 0$, $k \geq 3$) when the machines are dependent. There are similar counterexamples in our problem when at least two of the α_k's are positive (cf. Example 4.2), so it is noteworthy that Theorem 3.1 shows there are no such counterexamples when just one α_k is positive.

4. Geometric Discounting Case

We now consider the discount sequence in which, for some $\alpha > 0$ and $n \geq 1$,

$$\alpha_k = \alpha^{k-1} \qquad \text{if } k \leq n$$

$$= 0 \qquad \text{if } k > n.$$

When $\alpha < 1$, α can be interpreted as a traditional discount factor, and $n = \infty$ ("infinite horizon") has meaning. For, when $n = \infty$ and $\alpha < 1$, the expected payoff of every strategy is bounded (by $(1-\alpha)^{-1}$, e.g.) for every F. Whereas, when $n = \infty$ and $\alpha \geq 1$, the expected payoff of a variety of strategies may be infinite if $1-F_i(a^{-1}-\varepsilon)$ is large enough for i=1 and 2. For example, if (p_1,p_2) has uniform density on the unit square, then every strategy has infinite expected payoff when $\alpha \geq 1$ (using Machine i exclusively gives

$$\sum_1^{\infty} \alpha^{k-1} E p_i^{\,k} = \sum_1^{\infty} \alpha^{k-1}/(k+1) = \infty).$$

When $\alpha > 1$, it can be viewed as the growth factor for payoffs that one might encounter, for example, in gambling situations in which one's fortune rises disproportionately with one's successes. When $\alpha = 1$ we have a traditional nondiscounted problem. In case $\alpha \geq 1$ we assume $n < \infty$.

For the results of this section we assume that p_1 and p_2 are initially (and, therefore, also henceforth) independent.

We first consider $\alpha < 1$ and $n = \infty$. We note that at stage k the current discount sequence is a constant multiple of the original sequence, and so the problem is changed only by changes in F.

The expected payoff using only Machine i is

(1) $$V_i(\alpha) = E\sum_1^\infty \alpha^{k-1}p_i^{\,k}$$

$$= E\frac{p_i}{1-\alpha p_i},$$

for $i = 1,2$. The function V_i has a nice interpretation as a generating function (cf. Berry and Fristedt [2, Example 5.3]). In particular, it shares many characteristics of the moment generating function.

Define $V_{i|S}(\alpha)$ to be the generating function of p_i conditioned on a single success with Machine i; the corresponding conditional measure is $dF_{i|S}(x) = xdF_i(x)/Ep_i$. Then

(2) $$V_i(\alpha) = E\left[p_i + \alpha\sum_1^\infty \alpha^{k-1}p_i^{\,k+1}\right]$$

$$= Ep_i + \alpha Ep_i E\left[\frac{p_i^{\,2}}{1-\alpha p_i}\right]/Ep_i$$

$$= Ep_i + \alpha Ep_i V_{i|S}(\alpha),$$

a result that also follows easily from the definitions of V_i and $V_{i|S}$. Clearly,

(3) $$V_{i|S}(\alpha) \geq V_i(\alpha),\ \alpha\in(0,1).$$

The next result reduces the number of strategies that need be considered to just two: use Machine 1 exclusively, and use Machine 2 exclusively. It is similar to theorem 3.1 in this sense, and it too is a stay-on-a-winner rule.

Theorem 4.1. Assume $\underset{\sim}{A} = (1,\alpha,\alpha^2,\ldots)$, where $\alpha\in(0,1)$, and p_1 and p_2 are independent. An optimal strategy is to use Machine i if $V_i(\alpha) = \max\{V_1(\alpha), V_2(\alpha)\}$. Furthermore, it is uniquely optimal to use a single machine (i.e., never to switch) unless both machines are optimal intially and either F_1 or F_2 is a one-point distribution.

Proof. According to Theorem 2.12.1 of Dubins and Savage [4], we need only show that

$$V = \max_i V_i(\alpha)$$

is excessive, that is, the expected value of V under either initial choice is not greater than V itself. Two cases will be considered according to which machine is used first. Without loss of generality, assume $V_1(\alpha) \geq V_2(\alpha)$. Let τ denote the strategy that uses Machine 1 whenever the current expected value of $p_1/(1-\alpha p_1)$ is not less than that of $p_2/(1-\alpha p_2)$.

Suppose first that Machine 2 is used initially and τ followed thereafter. The total expected payoff of this strategy is

$$Ep_2 + \alpha Ep_2 \max\{V_1(\alpha),\ V_{2|S}(\alpha)\}. \tag{4}$$

If $V_{2|S}(\alpha) \geq V_1(\alpha)$ then (4) becomes

$$Ep_2 + \alpha Ep_2 V_{2|S}(\alpha) = V_2(\alpha),$$

in view of (2), which is not greater than $V_1(\alpha)$ by assumption. If $V_1(\alpha) \geq V_{2|S}(\alpha)$, then we need to show

$$Ep_2 + \alpha Ep_2 V_1(\alpha) \leq V_1(\alpha),$$

or equivalently,

$$\frac{Ep_2}{1 - \alpha Ep_2} \leq V_1(\alpha).$$

The function $x/(1 - \alpha x)$ is convex for $x \in [0,1]$, so Jensen's inequality applies to show that

$$\frac{Ep_2}{1 - \alpha Ep_2} \leq E\left[\frac{p_2}{1 - \alpha p_2}\right] = V_2(\alpha) \leq V_1(\alpha).$$

Now suppose Machine 1 is used first and τ followed thereafter. The total expected payoff of this strategy is

$$Ep_1 + \alpha Ep_1 \max\{V_{1|S}(\alpha), V_2(\alpha)\}. \tag{5}$$

Of course, if $V_{1|S}(\alpha) \geq V_2(\alpha)$ then (5) becomes

$$Ep_1 + Ep_1 V_{1|S}(\alpha) = V_1(\alpha),$$

in view of (2), or directly from the definition of τ. If $V_2(\alpha) \geq V_{1|S}(\alpha)$ then

$$Ep_1 + \alpha Ep_1 V_2(\alpha) \leq Ep_1 + \alpha Ep_1 V_1(\alpha)$$

$$\leq Ep_1 + \alpha Ep_1 V_{1|S}(\alpha) = V_1(\alpha),$$

from (3) and (2).

Therefore, V is excessive; so $V = V^*$ and every strategy (e.g., τ) which has expected payoff V is optimal.

The uniqueness conclusion in the theorem follows from the fact that (3) holds with equality if and only if F_i is a one-point distribution. □

<u>Example 4.1</u>. Suppose p_1 has a uniform density on $(0,1)$ and F_2 concentrates all its mass at 2/3; so that p_2 is known to be 2/3. Then

$$V_1(\alpha) = \sum_1^\infty \frac{\alpha^{k-1}}{k+1} = -\alpha^{-2}\log(1-\alpha) - \alpha^{-1},$$

and, of course,

$$V_2(\alpha) = \frac{2/3}{1 - 2\alpha/3}.$$

These are pictured in Figure 1.

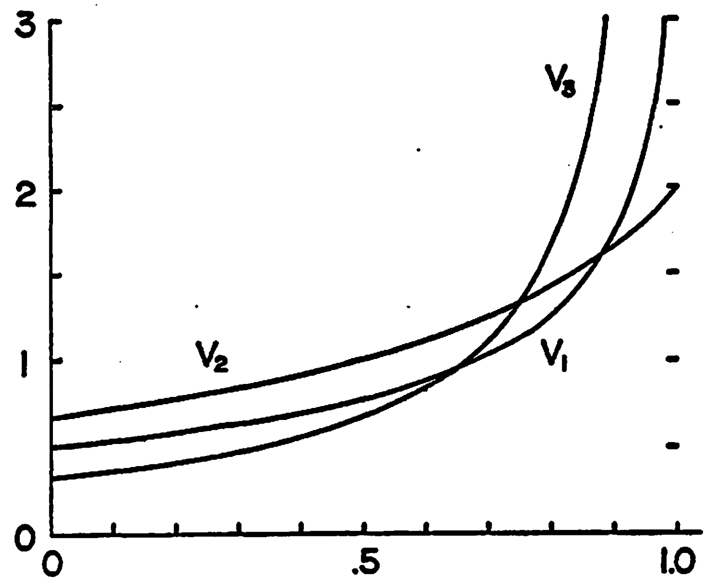

FIGURE 1
Comparison of Machines 1 and 2 in Example 3.1

Define α^* by $V_1(\alpha^*) = V_2(\alpha^*)$; for this example $\alpha^* \doteq .8834$. Machine 2 is optimal for $\alpha \leq \alpha^*$ and Machine 1 is optimal for $\alpha \geq \alpha^*$. It follows from Theorem 4.1 that for any $\alpha \in (0,1)$, there is an optimal strategy that never switches. However, since p_2 is known in this example, if $\alpha = \alpha^*$ then a switch from Machine 2 to Machine 1 after any number of successes is also optimal since both machines are optimal initially and F_2 is unchanged by outcomes on Machine 2. But a switch from Machine 1 to Machine 2 is not optimal since F_1 is changed by outcomes on Machine 1; in fact, $V_{1|S}(\alpha) > V_2(\alpha)$, for all $\alpha \in (0,1)$. □

The above example illustrates a phenomenon that holds more generally for the problem considered here (cf. Example 3.1), and for other two-armed bandit problems as well. When α is large, the prospect of future payoffs makes it worthwhile to use a machine about which little is known, even if this means sacrificing some immediate payoff. That is, when α

is large, the higher moments of F_1 and F_2 play an important role in the decision problem. When α is small, however, the higher moments are less important and the wise decision maker is reluctant to sacrifice early payoff. (It is somewhat curious in Example 4.1 that $V_2(\alpha) - V_1(\alpha)$ is actually increasing for small and moderate values of α.)

As an illustration of this phenomenon consider random variables on $[0,1]$ that have the largest variance for given mean: namely, random variables on $\{0,1\}$. Random variables with common mean, say μ, have generating functions equal at $\alpha = 0$, and all such generating functions are uniformly dominated by that of the random variable supported by $\{0,1\}$ with mass μ at 1. One such function,

$$V_3(\alpha) = \frac{1/3}{1 - \alpha}$$

is plotted in Figure 1; here $\mu = 1/3$. The advantage of a corresponding machine is evident for large α. From the opposite point of view, generating functions for machines with known characteristics provide a lower bound for machines with the same mean. So that if a machine has (expected) probability of success μ, then its generating function lies in the interval

$$\left[\frac{\mu}{1 - \mu\alpha}, \frac{\mu}{1 - \alpha}\right]$$

for any $\alpha \in [0,1]$.

Another point to be made from Figure 1 is that, since Theorem 4.1 applies as well for an arbitrary number of machines, the choice among the three machines with generating functions pictured can be made by choosing the machine with the largest $V(\alpha)$ and using that machine exclusively.

The next example illustrates another aspect of the relation between known machines and machines about which learning is possible, but its main purpose is to provide a counter-example to stay-on-a-winner rule in the independent case by going outside the geometric discounting case.

Example 4.2. Let F_1 and F_2 be as in Example 4.1, again with p_1 and p_2 independent, but now assume that

$$\underset{\sim}{A} = (10,\alpha,\alpha^2,\alpha^3,\ldots).$$

If $\alpha \leq \alpha^* \doteq .8834$, then it seems clear from the calculations in Example 4.1 that Machine 2 is optimal initially and henceforth. If α is slightly larger than .8834 (any number between .884 and .987 will do), then Machine 2 is still optimal initially since $\alpha_1 = 10$ is so large compared to the other α_k. However, if Machine 2 is successful initially, then, after normalizing, the new discount sequence is $(1,\alpha,\alpha^2,\ldots)$ and F_1 and F_2 are unchanged. Therefore, Example 4.1 applies and, since $\alpha > \alpha^*$, Machine 1 is now uniquely optimal. So, in this case, the known machine is used to reap an early benefit and the unknown machine is then used on the chance that it will provide some long-term benefits. □

We now consider the finite horizon case with arbitrary positive α. The total expected payoff using only Machine i is

$$\begin{aligned} V_i(\alpha,n) &= E\sum_1^n \alpha^{k-1} p_i{}^k \\ &= nP_{F_i}(p_i = \alpha^{-1}) + E\left[\frac{p_i}{1-\alpha p_i}(1-\alpha^n p_i{}^n)\,\middle|\, p_i \neq \alpha\right] P_{F_i}(p_i \neq \alpha^{-1}) \\ &= E\left[\frac{p_i}{1-\alpha p_i}(1-\alpha^n p_i{}^n)\right] \quad \text{if} \quad P_{F_i}(p_i = \alpha^{-1}) = 0. \end{aligned}$$

The next theorem says that an optimal strategy for the finite horizon case is similar to that for the infinite horizon case. We give the theorem without proof since it is very similar to the proof of Theorem 4.1; now, $V = \max\{V_1(\alpha,n),\ V_2(\alpha,n)\}$ is shown to be excessive.

Theorem 4.2. Assume $\underset{\sim}{A} = (1,\alpha,\alpha^2,\ldots,\alpha^{n-1},0,\ldots)$, where $\alpha > 0$ and $1 \leq n < \infty$, and p_1 and p_2 are independent. An optimal strategy is to use Machine i if $V_i(\alpha,n) = \max\{V_1(\alpha,n), V_2(\alpha,n)\}$. Furthermore, it is uniquely optimal to use a single machine unless F_1 and F_2 are the same one-point distribution.

Example 4.3. The determinants of the value of alternative machines follow the expected patterns. Let Machine i be characterized by a beta prior (see Example 3.1), $i = 1,2$. Then, it is straightforward to show the V_i increases with α and a_i and decreases with b_i. Furthermore, if $Ep_i = a_i/(a_i + b_i)$ is fixed while $a_i + b_i$ is decreased, then V_i increases. Again, machines with properties that are dimly understood are preferred since they offer a greater opportunity for long-term survival. This is especially important if α is large. □

5. Conclusions

The termination of a two-armed bandit problem after the first unsuccessful outcome has similar implications both for situations in which outcomes in each period are valued and for contexts in which only terminal rewards are of consequence. If the trials on the two machines are independent, the stay-on-a-winner rule holds, as in traditional models of this type. For the terminal rewards situation, staying with a winner is also optimal when the trials are interdependent. Both situations are generalizable to an arbitrary number of machines.

In the conventional two-armed bandit models, the preference for machines associated with little prior information derives from the potential for learning through experience about an uncertain alternative and then staying with it if one's experiences are favorable and switching to some other policy if the outcomes are sufficiently unfavorable. While no such adaptation is possible when adverse outcomes terminate the decision problem, "loose" priors are preferred, relatively speaking, since they offer greater prospects for long-term survival.

References

[1] D.A. Berry, "A Bernoulli two-armed bandit," Ann. Math. Statist. 43 (1972) 871-897.

[2] D.A. Berry and B. Fristedt, "Bernoulli one-armed bandits--arbitrary discount sequences," Ann. Statist. 7 (1979) 1086-1105.

[3] R.N. Bradt, S.M. Johnson, and S. Karlin, "On sequential designs for maximizing the sum of n observations," Ann. Math. Statist. 27 (1956) 1060-1074.

[4] L.E. Dubins and L.J. Savage, Inequalities for Stochastic Processes: How to Gamble If You Must (Dover, New York, 1976).

[5] J. Fabius and W.R. van Zwet, "Some remarks on the two-armed bandit," Ann. Math. Statist. 41 (1970) 1906-1916.

[6] D. Feldman, "Contributions to the 'two-armed bandit' problem," Ann. Math. Statist. 33 (1962) 847-856.

[7] J.C. Gittins, "Bandit processes and dynamic allocation indices," J. Royal Statist. Soc., Series B 41 (1979) no. 2.

[8] V.M. Joshi, "A conjecture of Berry regarding a Bernoulli two-armed bandit," Ann. Statist. 3 (1975) 189-202.

[9] W. Thompson, "On the likelihood that one unknown probability exceeds another in view of the evidence of two samples, Biometrika XXV (1933) 285-294.

[10] W.K. Viscusi, Employment Hazards: An Investigation of Market Preformance (Harvard University Press, Cambridge, 1979).

[11] W.K. Viscusi and R. Zeckhauser, "Environmental policy change under uncertainty," J. Environmental Economics and Management, 3 (1976) 97-112.